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RATIONALITY GROUPS IN PRESCRIBED DOMAINS

BY S. EPSTEEN

1. Introduction. Just as the galoisian group of an algebraic equation reflects the essential characteristics of the equation, so does the rationality group reflect the essential characteristics of a linear homogeneous differential equation.

In studying an algebraic equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

by the Galois' method, we first specify a domain of rationality, and a certain number of the n! permutations of the roots $x_1, \cdots x_n$, constitute the galoisian group of the equation.

Domain of rationality. A domain of rationality is a set of numbers whose sum, difference, product, and quotient (exclusive of the zero divisor) always give a number of the set. Frequently, the domain of rational numbers, r, is taken. One also uses the domain $c = r[i]$, that is, the domain r with $i = \sqrt{-1}$ adjoined; this gives numbers of the form $u + iv$, u and v denoting numbers of r. Likewise, the domain of real numbers, R, and the domain $C = R[i]$, are used. In some instances, when one wishes to state a theorem for any arbitrary domain, the symbol D is convenient.

Galois' group. The group G of an equation in a given domain D is the totality of substitutions having the following double property:

a) Every rational function of the roots, numerically unaltered by all the substitutions of the group, has its numerical value in D; and

b) Every rational function of the roots, having its numerical value in D, is numerically unaltered by the substitutions of G.

In studying a linear homogeneous differential equation

$$K(y) \equiv \frac{d^n y}{dx^n} + k_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_n(x) \cdot y = 0$$

by the Picard-Vessiot method, we first specify a derivative domain of rationality, and a certain sub-group of the general linear homogeneous group in n variables and n^2 parameters

$$L_{n^2}: \quad \eta_{j_1} = \sum_{j_2} a_{j_1 j_2} y_{j_2} \qquad (j = 1 \cdots n)$$

constitutes the rationality group of the equation. Here $y_1, \cdots y_n$ denote a fundamental system of K, and $|a_{j_1 j_2}| \neq 0$.

Derivative domain of rationality. A derivative domain of rationality is composed of one of the foregoing domains r, c, R, C, and any arbitrary number of functions analytic in a specified region of the plane of the complex variable x. The operations of addition, subtraction, multiplication, division (exclusive of the divisor zero), and *differentiation* always reproduce elements of the domain. We denote by $R[x]$ and $C[x]$ the domains R and C with the independent variable x adjoined; * by $R[x, f(x)]$ the domain $R[x]$ with $f(x)$ adjoined. In some instances, when one wishes to state a theorem for any arbitrary domain, the symbol $D\}x\{$ is convenient.

A domain $D\}x\{$ being given, a number or a function is said to be *rational* if it is in this domain, and *irrational* if it is not in this domain.

Group of Rationality. By the rationality group G of a linear homogeneous differential equation of order n, in a domain $D\}x\{$, is meant the totality of linear homogeneous substitutions of non-vanishing determinant on a fundamental system of integrals $y_1, \cdots y_n$:

$$\eta_{j_1} = \sum_{j_2} a_{j_1 j_2} y_{j_2}, \qquad (j = 1 \cdots n)$$

which possess the property known as the Picard-Vessiot Double Theorem: †

A) Every rational differential function of $y_1, \cdots y_n$:

$$F(x, y_1 \cdots y_n, y_1' \cdots y_n', \cdots)$$

which is numerically invariant under the substitutions of G has its numerical value, $r(x)$, in the domain $D\}x\{$; and inversely,

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* The necessity of explicitly adjoining the independent variable x was first pointed out by A. Loewy, *Mathematische Annalen*, vol. 59, p. 436. A domain may contain functions of x without containing x itself, as for example $R[e^x]$.

† Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, II, 1, p. 71.

B) Every rational differential function of $y_1, \cdots y_n$:

$$F(x, y_1 \cdots y_n, \; y_1' \cdots y_n', \cdots)$$

which has a numerical value $r(x)$ in the domain $D\{x\}$ is invariant under the substitutions of G.

The literature of the Picard-Vessiot Theory is now quite extensive, but any one familiar with the elements of the Galois' Theory of Algebraic Equations and Lie's Theory of Continuous Groups will find it easy to read the summaries in Picard's *Traité d'Analyse*, vol. 3, last chapter; Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, II, 1, pp. 1–125; and Klein's *Höhere Geometrie*, vol. 2, pp. 265–302.

2. The Equations and Domains. In the *Vorlesungen über Höhere Geometrie*, vol. 2, p. 300, Klein points out the desirability of giving examples of differential equations with their rationality groups for specified domains of rationality. In the present paper this suggestion is carried out for the following equations and domains:

I. Equation of the Second Order, with constant coefficients:

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0,$$

for the domains of rationality:

a) R; *b*) C; *c*) $R[x]$; *d*) $C[x]$; *e*) $R[x, e^x]$; *f*) $C[x, e^x] \equiv C[x, \sin x]$.

II. Hypergeometric (Gauss') equation:

$$\frac{d^2y}{dx^2} + \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1-x)} \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)} y = 0,$$

for the domains $R[x]$ and $C[x]$.

III. Moduli of Periodicity of Elliptic Integrals of the First Kind:

$$\frac{d^2y}{dx^2} + \frac{1-2x}{x(1-x)} \frac{dy}{dx} - \frac{1}{4x(1-x)} y = 0,$$

for the domains $R[x]$ and $C[x]$.

IV. Legendre's Polynomial of order k:

$$\frac{d^2y}{dx^2}+\frac{1-2x}{x(1-x)}\frac{dy}{dx}+\frac{k(k+1)}{x(1-x)}y=0 \quad (k=\text{positive integer});$$

domains $R[x]$, $C[x]$.

V. Bessel's Equation:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0;$$

domains: *a*) $R[x]$; *b*) $C[x]$; *c*) $R[x, \sin x]$; *d*) $C[x, \sin x]$.

VI. Cauchy's Equation:

$$\frac{d^2y}{dx^2}+\frac{a_1}{x}\frac{dy}{dx}+\frac{a_2}{x^2}y=0;^*$$

domains: *a*) R; *b*) C; *c*) $R[x]$; *d*) $C[x]$; *e*) $C[x, \log x]$.

The solution of I, *d*, has been given, in the main, by Vessiot.† All the other examples I believe to be new.

3. Algebraic Subgroups of L_{2^2}. The equations investigated in this paper are all of the second order. Following the outline of Vessiot‡ we write down the algebraic subgroups of the general linear homogeneous group L_{2^2} in 2 variables and 2^2 parameters,§ whose infinitesimal transformations are

$$L_2^2: \qquad y_1p_1, \quad y_1p_2, \quad y_2p_1, \quad y_2p_2; \qquad \left(p_i \equiv \frac{\partial f}{\partial y_i}\right),$$

together with their differential invariants of first order.

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* Our results apply equally well to Cauchy's equation in the form

$$y''+\frac{a_1}{x-a}y'+\frac{a_2}{(x-a)^2}y=0.$$

† E. Vessiot, *Annales de l'Ecole Normale Supérieure*, vol. 9 (1892), pp, 248–9, Vessiot found the group of the equation of order n with constant coefficients, but did not specify a domain of rationality. Applied to $n=2$, his determination is that of §5 *d*; cases 1 and 3.

‡ Vessiot, loc. cit., p. 249. F. Marotte, *Comptes Rendus*, vol. 124 (1897), pp. 608–610.

§ Vessiot, loc. cit., p. 258.

	GROUP	DIFFERENTIAL INVARIANTS
A.	$y_1p_2,\ y_1p_1 - y_2p_2,\ y_2p_1$ (special linear group)	$y_1y_2' - y_1'y_2 \equiv \Delta$
B.	$y_1p_1,\ y_1p_2,\ y_2p_2$	$\frac{y_1'}{y_1}$
C.	$y_1p_2,\ y_1p_1 + y_2p_2$	$\frac{y_1'}{y_1},\ \frac{\Delta}{y_1^2} \equiv \frac{d}{dx}\left(\frac{y_2}{y_1}\right) \equiv w$
D.	$y_1p_1,\ y_2p_2$	$\frac{y_1'}{y_1},\ \frac{y_2'}{y_2}.$
E.	$ay_1p_1 + by_2p_2,\ y_1p_2$	$\frac{y_1'}{y_1},\ \frac{\Delta^a}{y_1^b}$
F.	$y_1p_1 + y_2p_2$	$\frac{y_1'}{y_1},\ \frac{y_2'}{y_2},\ \frac{y_2}{y_1} \equiv v$
G.	y_1p_2	$y_1,\ \Delta$
H.	$ay_1p_1 + by_2p_2$	$\frac{y_1'}{y_1},\ \frac{y_2'}{y_2},\ \frac{y_2^a}{y_1^b}.$

Since the listed invariants must be rational, the parameters a and b in E and H are necessarily integral numbers.

4. The Resolvents.* For an equation of second order,

$$K(y) = \frac{d^2y}{dx^2} + k_1(x)\,\frac{dy}{dx} + k_2(x)y = 0,$$

the invariant Δ satisfies the resolvent

$$R_A: \qquad \Delta' + k_1\,\Delta = 0.$$

It is best to replace the invariant y_1'/y_1 of the groups B, C, D, E, F, (G), H by

$$u_1 = \frac{y_1'}{y_1} - \frac{1}{2}\,\frac{y_1y_2'' - y'\,y_2}{y_1y_2' - y_1'y_2} = \frac{y_1'}{y_1} + \frac{1}{2}\,k_1.$$

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* Vessiot, loc. cit , p. 217 calls this the *transformed*. I prefer the term *resolvent* by analogy with the Galois' Theory of Equations.

This function satisfies the resolvent Riccati equation

$$R_B: \qquad u' + u^2 + I = 0,$$

where

$$I = k_2 - \frac{1}{4} k_1^2 - \frac{1}{2} k_1'.$$

The invariant $v = y_2/y_1$ of the group F satisfies the resolvent

$$R_F: \qquad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + I = 0.$$

From the latter it is easy to see that the invariant

$$w = \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = v'$$

of the group C, satisfies the resolvent

$$R_C: \qquad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + I = 0.$$

If A is the (rationality) group of the equation $K(y) = 0$ in a domain $D\}x\{$, then by the Picard-Vessiot Double Theorem, Δ, the invariant of the group A, must have a rational value. Therefore the resolvent R_A has a rational integral; and inversely.

Similarly, if $K(y)$ has for its rationality group either B, C, D, E, F, (G) or H, the resolvent R_B has a rational integral; and inversely. Indeed, if the group of the equation is D or one of its subgroups, R_B has *two* rational integrals,

$$u_1 = \frac{y_1'}{y_1} + \frac{1}{2} k_1, \qquad u_2 = \frac{y_2'}{y_2} + \frac{1}{2} k_2;$$

and inversely.

If $K(y)$ has C for its rationality group, then R_C has a rational integral; and if it has F for its group, then R_F has a rational integral; and inversely.

Our procedure will be: (1) to set up the resolvents for each equation I, II, III, IV, V, VI; (2) to determine which have or have not rational integrals (for the domains listed in §1), and thus: (3) the rationality groups of the equations for these domains.

The notation that will be used is an obvious one, R_A^I denoting the resolvent R_A with the value of I taken from equation I, R_B^{III} the resolvent R_B with the value of I taken from the equation III, and so forth.

5. Equation with Constant coefficients. For the equation I we have, $I = a_2 - \frac{a_1^2}{4}$, and the resolvents become

$$R^I_A: \quad \Delta' + a_1 \Delta = 0,$$

$$R^I_B: \quad u' + u^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0,$$

$$R^I_C: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0,$$

$$R^I_F: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + \left(a_2 - \frac{a_1^2}{4}\right) = 0.$$

(**a**) *Let the domain of rationality be R, the totality of real numbers.*

Case 1. $a_1 \neq 0$, $I > 0$. Since $e^{-a_1 x}$ is irrational (not in the domain R), the resolvent R^I_A has an irrational integral $\Delta = e^{-a_1 x}$; this excludes the groups A and G. We have for integrals of R^I_A

$$u_1 = i\sqrt{I}, \quad u_2 = -i\sqrt{I},^{*} \quad u_3 = -\sqrt{I}\tan x\sqrt{I}, \ (i = \sqrt{-1}),$$

none of which is rational. Therefore the rationality group of the equation I cannot be B, C, D, E, F, (G), H, and must be the general linear homogeneous group L_{2^2}.

Illustrative equation :

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

Case 2. $a_1 = 0$, $I > 0$. In this case R^I_A has a rational integral, $\Delta = 1$, and the group of the equation is the special linear group A.

Illustrative equation :

$$\frac{d^2y}{dx^2} + 5y = 0.$$

Case 3. $a_1 \neq 0$, $I = 0$. The resolvent R^I_B has a rational integral $u_1 = 0$, and the resolvant R^I_C has a rational integral $w = 1$. Therefore the rationality group of the equation is the group C.

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[*] Unlike the case in linear equations, u_1 and u_2 must be regarded as distinct; and indeed, they lead to different integrals of I,

$$y_1 = e^{\frac{-a_1 x}{2}} e^{u_1 x}, \quad y_2 = e^{\frac{-a_1 x}{2}} e^{u_2 x}.$$

Illustrative equation:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.^*$$

Case 4. $a_1 = 0$, $I = 0$. The group of the equation in the domain R is G.

Illustrative equation:

$$\frac{d^2y}{dx^2} = 0.$$

Case 5. $a_1 \neq 0$, $I < 0$. The resolvent R_B^I has two rational integrals,

$$u_1 = \sqrt{-I}, \qquad u_2 = -\sqrt{-I}.$$

The group of the equation must be D, F, or H. One can exclude F by noting that R_F^I does not have an integral in the domain R. We could set up the resolvent for the invariant $\frac{y_2^a}{y_1^b}$ of H and find under what conditions it has a rational integral. I have not completed this calculation on account of its extreme length; however, the result may be obtained in the following simple (though irregular) manner. The integrals of I are

$$y_1 = e^{\left(-\frac{a_1}{2} + \sqrt{-I}\right)x}, \qquad y_2 = e^{\left(-\frac{a_1}{2} - \sqrt{-I}\right)x},$$

and it can be seen that

$$\frac{y_2^a}{y_1^b} = 1$$

when $a = k(a_1 - 2\sqrt{-I})$, $b = k(a_1 + 2\sqrt{-I})$, k denoting any number in the domain R. Therefore, if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are integers, the group of the equation is H, while if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are not integers, the group of the equation is D.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = 0, \text{ group } H \text{ with } a = 1, \quad b = 9.$$

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* Two integrals are of course, $y_1 = e^{4x}$, $y = xe^{4x}$; and it is interesting to actually verify that the invariants of the other group (§3) are irrational; thus an invariant of F, $y_2/y_1 = x$, is not in the domain R.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \pi \frac{dy}{dx} + y = 0, \quad \text{group } D.$$

Case 6. $a_1 = 0$, $I < 0$. The resolvent R_A^I has the rational integral $\Delta = 1$, and the resolvent R_B^I has two rational integrals

$$u_1 = \sqrt{-I}, \quad u_2 = -\sqrt{-I}.$$

The greatest subgroup common to A and D, viz:

H': $$y_1p_1 - y_2p_2$$

is the group of the equation. H' is the group H, with $a = -b = 1$.

Illustrative equation: $$\frac{d^2y}{dx^2} - 4y = 0.$$

b). *Let the domain of rationality be C, the totality of complex numbers.* In this domain we do not need to distinguish between $I < 0$ and $I > 0$, since in either event, $\sqrt{\pm I}$ is rational (lies in the domain C).

Case 1. $a_1 \neq 0$, $I = a_2 - a_1^2/4 \neq 0$. The resolvent R_B^I has two rational integrals

$$u_1 = \sqrt{-I}, \qquad u_2 = -\sqrt{-I}.$$

As in a), case 5, the group of the equation is H if $k(a_1 - 2\sqrt{-I})$ and $k(a_1 + 2\sqrt{-I})$ are integers, and it is the group D if these numbers are not integers. The illustrative equations of a) case 5 can serve here also.

Case 2. $a_1 = 0$, $I \neq 0$. As in a), case 6, the group of the equation is $H' \equiv y_1p_1 - y_2p_2$. Same illustrative equation as in a), case 6.

Case 3. $a_1 \neq 0$, $I = 0$. The resolvents R_B^I and R_C^I have each one rational integral, $u_1 = 0$, $w_1 = 1$; therefore the group of the equation is C.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 2(1+i)\frac{dy}{dx} + 2iy = 0.$$

Case 4. $a_1 = 0$, $I = 0$. Same as a), case 4.

c) *Let the domain of rationality be $R[x]$, the totality of rational functions of x with real coefficients.*

Case 1. $a_1 \neq 0$, $I > 0$. Same as a), case 1.

Case 2. $a_1 = 0$, $I > 0$. Same as a), case 2.

Case 3. $a_1 \neq 0$, $I = 0$. The resolvent R_B^I has two rational integrals, $u_1 = 0$, $u_2 = 1/x$ in the domain $R[x]$, and the resolvent R_F^I has a rational integral $v = x$; therefore the group of the equation is F.*

Case 4. $a_1 = 0$, $I = 0$. The resolvent R_A^I has a rational integral $\Delta = 1$, the resolvent R_B^I has two rational integrals $u_1 = 0$, $u_2 = 1/x$ and R_F^I has a rational integral $v = x$; there is no subgroup common to A and F except the identical transformation. Therefore the group of the equation is the *Identity*.

Case 5. $a_1 \neq 0$, $I < 0$. Same as a), case 5.

Case 6. $a_1 = 0$, $I < 0$. Same as a), case 6.

d) *Let the domain of rationality be $C[x]$, the totality of rational functions of x with real or complex coefficients.*

Case 1. $a_1 \neq 0$, $I \neq 0$. Same as b), case 1.

Case 2. $a_1 = 0$, $I \neq 0$. Same as b), case 2.

Case 3. $a_1 \neq 0$, $I = 0$. Same as c), case 3.

Case 4. $a_1 = 0$, $I = 0$. Same as c), case 4.

e) *Let the domain of rationality be $R[x, e^x]$.* In this domain the resolvent R_A^I has the rational integral $\Delta = e^{-a_1 x}$, and the rationality group of the equation must be A or one of the subgroups.

Case 1. $I > 0$. Since R_B^I does not have a rational integral, the group of the equation is A.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0.$$

Case 2. $I = 0$. The resolvent R_B^I has two rational integrals, $u_1 = 0$, $u_2 = 1/x$ and the resolvent R_F^I has two rational integrals $v_1 = x$, $v_2 = -1/x$. The only subgroup common to A and F is the *Identity*, which must be the group of the equation.

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* The resolvent R_C^I has a rational integral $w = 1$, but this is also a consequence of the fact that $w = dv/dx$.

Illustrative equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

Case 3. $I < 0$. The resolvent R_A^I has one rational integral $\Delta = e^{-a_1 x}$;

R_B^I has two rational integrals $v_1 = \frac{1}{2\sqrt{-I}}\, e^{2x\sqrt{-I}}, \quad v_2 = \frac{1}{2\sqrt{-I}}\, e^{-2x\sqrt{-I}}$.

The group of the equation is the *Identity*.

f) *Let the domain of rationality be* $C[x, e^x] \equiv C[x, \sin x]$. In this domain the group of the equation with constant coefficients is the *Identity*. In domains larger than $C[x, e^x]$, as for instance $C[x, e^x, f(x)]$ the group of the equation I is always *Identity*.

In smaller domains, such as $C[e^x]$, $R[e^x]$, the domain of rational numbers, etc., certain slight modifications of the above investigation are needed, and are easily made.

6. The Hypergeometric Equation. For the equation II we have

$$I_{II} = \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2},$$

where

$$h_1 = k_1 = -\,a\beta - \tfrac{1}{2}\,\gamma(\gamma - a - \beta - 1),$$

$$h_2 = -\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{2},$$

$$k_2 = -\left(\frac{\gamma - a - \beta - 1}{2}\right)^2 - \frac{\gamma - a - \beta - 1}{2}.$$

The principal resolvents from which the rationality groups will be deduced are:

$$R_A^{II}\colon \quad \frac{d\Delta}{dx} + \frac{\gamma - (a + \beta + 1)x}{x(1-x)}\,\Delta = 0,$$

$$R_B^{II}\colon \quad \frac{du}{dx} + u^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0,$$

$$R_C^{II}\colon \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0,$$

$$R_F^{II}\colon \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + \frac{h_1}{x} + \frac{h_2}{x^2} + \frac{k_1}{1-x} + \frac{k_2}{(1-x)^2} = 0.$$

The relations between the solutions of R^{II}_B, R^{II}_C, R^{II}_F are useful:

$$w = e^{-2\int u dx}, \quad v = \int w dx.$$

We must find under what conditions the resolvents have rational integrals.

a) R^{II}_A will have its integral

$$\Delta - \frac{(1-x)^{\gamma-(a+\beta+1)}}{x^{\gamma}}$$

rational, if the exponents $\gamma - (a + \beta + 1)$ and γ are integers; hence it will be rational if

$$a + \beta = \text{integer, and,} \quad \gamma = \text{integer.}$$

These integers may be positive or negative.

b) It is known[*] that R^{II}_B, has a rational integral

$$u = \frac{a_1}{x} \frac{a_2}{1-x} + \frac{g'(x)}{g(x)},$$

provided that one of the four quantities

$$a, \beta, \gamma - a, \gamma - \beta$$

is a negative integer. Here a_1 and a_2 have the values

$$a_1' = \frac{\gamma}{2}, \qquad a_1'' = 1 - \frac{\gamma}{2},$$

$$a_2' = \frac{\gamma - a - \beta - 1}{2}, \qquad a_2'' = -1 - \frac{\gamma - a - \beta - 1}{2},$$

and $g(x)$ is a polynomial (without multiple roots) of degree n, where n has one of the values

$$-a, -\beta; \; -a + 1, \; -\beta + 1; \; a - \gamma; \; \beta - \gamma; \; a - \gamma + 1, \; \beta - \gamma + 1.$$

c) Since $w = e^{-2\int u dx}$, we have

$$w = \frac{(1-x)^{2a_2}}{x^{2a_1} g^2},$$

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[*] E. Beke, Die Irreducibilität der homogenen linearen Differentialgleichungen, *Mathematische Annalen*, vol. 45 (1894), p. 287. See also Picard's *Traité d'Analyse*, 2nd edition, vol. 3, p 561.

Since the part of Beke's paper (§6) which bears directly on the present investigation is valid in the domains $R[x]$ and $C[x]$, the results here deduced are valid in the same domains and the integral w of R''_C will be rational if u is a rational function and if $2a_1$ and $2a_2$ are integers, that is, if

$$u = \text{rat.}\, f(x) \quad \text{and} \quad \gamma = \text{integer}, \quad \alpha + \beta = \text{integer}.$$

By comparing with (a) we are led to an interesting conclusion: if R''_A and R''_B have each a rational integral, then R''_C has also a rational integral. The group common to A, B and C being G, we see that: *if R''_A and R''_B have each a rational integral, the group of the equation will be G.*

f) To determine whether R''_F has a rational integral, we consider that

$$v = \int w dx,$$

and

$$w = \frac{(1-x)^{2a_2}}{x^{2a_1}\, g^2},$$

where a_1 and a_2 are known integers and g a known polynomial. Therefore, for any particular problem, the determination of w is a straightforward process which can be completed in a finite number of steps.

Let the domain of rationality be either $R[x]$ or $C[x]$.

Case 1. The rationality group will be the general linear homogeneous group L_{2^2} if each of the following conditions is unfulfilled:

z) $\alpha + \beta = \text{integer}$, $\gamma = \text{integer}$;

y) $\alpha = \text{negative integer}$;

x) $\beta = \text{negative integer}$;

w) $\gamma - \alpha = \text{negative integer}$;

v) $\gamma - \beta = \text{negative integer}$.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{6-9x}{2x(1-x)} - \frac{3}{x(1-x)}\, y = 0.$$

Case 2. The rationality group of the equation will be the special linear group if

$$z) \quad \alpha + \beta = \text{integer}, \; \gamma = \text{integer},$$

and if each of the other conditions of case 1, (y, x, w, v) is unfulfilled.

Illustrative equation:

$$\frac{d^2y}{dx^2}+\frac{1-5x}{x(1-x)}-\frac{15}{4x(1-x)}\,y=0.$$

Case 3. If α is a negative integer, and if every other condition of case 1, (z, x, w, v) is unfulfilled, R_B^{II} will have exactly one rational integral, and the group of the equation will be B.

Similar remarks apply to the possibilities shown in the following table; each of the equations has B for its group.

NEGATIVE INTEGER	UNFULFILLED	ILLUSTRATIVE EQUATION
α	$z,\ x,\ w,\ v$	$y''+\frac{4+x}{2x(1-x)}\,y'+\frac{10y}{x(1-x)}=0$
β	$z,\ y,\ w,\ v$	$y''+\frac{4+x}{2x(1-x)}\,y'+\frac{10y}{x(1-x)}=0$
$\gamma-\alpha$	$z,\ y,\ x,\ v$	$y''+\frac{11x-8}{2x(1-x)}\,y'-\frac{5y}{x(1-x)}=0$
$\gamma-\beta$	$z,\ y,\ x,\ v$	$y''+\frac{11x-8}{2x(1-x)}\,y'-\frac{5y}{x(1-x)}=0$

The first and second illustrations of this table are purposely taken alike, and similarly for the third and fourth. In the first and second we have

$$\gamma=2,\quad \alpha+\beta+1=-\tfrac{1}{2},\quad \alpha\beta=-10,$$

whence,

$$\alpha=-4,\ \beta=\tfrac{5}{2},\qquad \gamma=2,$$

the first illustration; or

$$\alpha=\tfrac{5}{2},\quad \beta=-4,\quad \gamma=2,$$

the second illustration.

Case 4. Let $\alpha=$ negative integer, $\beta=$ negative integer (different from α), and $\gamma\neq$ integer. R_B^{II} has now two rational integrals, and the group of the equation is D. The same remarks apply to the possibilities shown in the following table:

NEG. INTEGERS (DIFFERENT)	NOT INTEGER	ILLUSTRATIVE EQUATION
$\alpha, \quad \beta$	γ	$y'' + \dfrac{5+8x}{2x(1-x)}\, y' - \dfrac{6}{x(1-x)}\, y = 0$
$\alpha, \quad \gamma - \alpha$	β	$y'' - \dfrac{6+5x}{2x(1-x)}\, y' + \dfrac{5}{2x(1-x)}\, y = 0$
$\alpha, \quad \gamma - \beta$	β	$y'' + \dfrac{3-5x}{2x(1-x)}\, y' + \dfrac{5}{2x(1-x)}\, y = 0$
$\beta, \quad \gamma - \alpha$	α	$y'' + \dfrac{3-5x}{2x(1-x)}\, y' + \dfrac{5}{2x(1-x)}\, y = 0$
$\beta, \quad \gamma - \beta$	α	$y'' - \dfrac{6+5x}{2x(1-x)}\, y' + \dfrac{5}{2x(1-x)}\, y = 0$
$\gamma - \alpha, \gamma - \beta$	γ, and $\alpha \neq$ neg. integer $\beta \neq$ neg. integer	$y'' + \dfrac{1-10x}{2x(1-x)}\, y' - \dfrac{15}{4x(1-x)}\, y = 0$

We shall not stop here to enumerate the large additional number of special cases that may arise. By following out the method of this section, it is possible to determine the group of any other given hypergeometric equation in the domains $R[x]$ and $C[x]$. We give two important instances in §7 and §8.

In the event of R_B^H having two rational integrals a special investigation is needed. Ordinarily, the group of the equation will be D, but if, in addition, $v = y_2/y_1$ is rational, the group of the equation is F. However, it may happen that the group of the equation is H. Let $t = y_2^a/y_1^b$ be the third invariant of H, then

$$\frac{t'}{t} = a\,\frac{y_2'}{y_2} - b\,\frac{y_1'}{y_1} = au_2 - bu_1 + \frac{ak_1}{2} - \frac{bk_1}{2},$$

and

$$t = e^{\int\left(au_2 - bu_1 + \frac{ak_1}{2} - \frac{bk_1}{2}\right)dx} \quad ;$$

if now, this value lies in the domain of rationality, the group of the equation is H.

7. The Equation for the Moduli of Periodicity of the Elliptic Integrals of the First Kind.

The equation III is a hypergeometric equation for which

$$\alpha = \beta = \tfrac{1}{2}, \quad \gamma = 1.$$

The only condition of §6, case 1, which is fulfilled is z. Therefore the resolvent R_A^{III} has a rational integral

$$\Delta = \frac{1}{x(1-x)},$$

and none of the other resolvents have rational integrals (indeed, the fact R_B^{II} does not have a rational integral is decisive, for it excludes groups B, C, D, E, F, G, H). The rationality group of the equation III is A in the domains $R[x]$ and $C[x]$.

8. Legendre's Polynomial of order k. Number IV is a hypergeometric equation for which

$$a = k+1, \quad \beta = -k, \quad \gamma = 1.$$

Here we have fulfilled the conditions

z) $a + \beta =$ integer, $\gamma =$ integer,

x) $\beta =$ negative integer,

w) $\gamma - a =$ negative integer.

(z) shows that R_A^{IV} has a rational integral, while (x) and (w) show that R_B^{IV} has two rational integrals. It follows (§6, c) that R_C^{IV} has two rational integrals. The rationality group of the equation is therefore the sub-group common to A, D, C, that is, the *Identity*.

9. Bessel's Equation. For the equation V we have

$$I = 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right),$$

and the corresponding resolvents are

$$R_A^V: \quad \Delta' + \frac{1}{x}\Delta = 0,$$

$$R_B^V: \quad u' + u^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0,$$

$$R_C^V: \quad \left(\frac{-w'}{2w}\right)' + \left(\frac{-w'}{2w}\right)^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0,$$

$$R_F^V: \quad \left(\frac{-v''}{2v'}\right)' + \left(\frac{-v''}{2v'}\right)^2 + 1 + \frac{1}{x^2}\left(\frac{1}{4} - n^2\right) = 0.$$

The Integral of R^V_A. The resolvent R^V_A has for an integral

$$\Delta = \frac{1}{x}.$$

The Integrals of R^V_B. A solution of the resolvent R^V_B has $x = 0$ for a pole. If it has another pole, $x = a$, one can see from the form of the resolvent that the coefficient of $(x - a)^{-1}$ in u, must be unity. Therefore the solution must be of the form

$$u_1 = i + \frac{h_1}{x} + \frac{g'(x)}{g(x)} \quad \text{or} \quad u_2 = -i + \frac{h_2}{x} + \frac{g'(x)}{g(x)},$$

where the g's are polynomials.

By substituting in R^V_B we find, first of all, that h_1 and h_2 can have either of two values

$$h'_1 = h'_2 = \tfrac{1}{2} + n,$$
$$h''_1 = h''_2 = \tfrac{1}{2} - n.$$

The substitution of u_1 and u_2 in R^V_B gives us, for the determination of g

$$xg'' + (2ix + 1 + 2n)\, g' + i(1 + 2n)\, g = 0,$$
$$xg'' + (2ix + 1 - 2n)\, g' + i\,(1 - 2n)\, g = 0,$$
$$xg'' + (-2ix + 1 + 2n)\, g' - i(1 + 2n)\, g = 0,$$
$$xg'' + (-2ix + 1 - 2n)\, g' - i(1 - 2n)\, g = 0.$$

If p is the degree of g, the highest power of x in these equations gives us

$$2ip + i(1 + 2n) = 0,$$
$$2ip + i(1 - 2n) = 0,$$
$$-2ip - i(1 + 2n) = 0,$$
$$-2ip - i(1 - 2n) = 0.$$

If g exists at all, p is an integer, and therefore $2n$ is an odd integer. The other powers of x in the equations for g, equated to zero, will give us the coefficients of the polynomials, step by step. In this manner we reach the conclusion that *the resolvent* R^V_B *has two rational integrals*

$$u_1 = i + \frac{\frac{1}{2} + n}{x} + \frac{g'_1(x)}{g_1(x)}, \qquad u_2 = -i + \frac{\frac{1}{2} + n}{x} + \frac{g'_2(x)}{g_2(x)};$$

or

$$u_1 = i + \frac{\frac{1}{2} - n}{x} + \frac{g_3'(x)}{g_3(x)}, \qquad u_2 = -i + \frac{\frac{1}{2} - n}{x} + \frac{g_4'(x)}{g_4(x)},$$

where the g's are determinable polynomials, if 2n is an odd integer.

The Integrals of R_C^V. Since $w = e^{-2\int u dx}$ (§4), u denoting a solution of R_B^V, we have

$$w_1 = e^{-2ix} \frac{x^{-1 \pm 2n}}{g^2} = \frac{(\cos 2x - i \sin 2x)x^{-1 \pm 2n}}{g^2},$$

$$w_2 = \frac{(\cos 2x + i \sin 2x)x^{-1 \pm 2n}}{g^2},$$

provided that g exists; that is, provided $2n$ is an odd integer. In particular, if $2n = \pm 1$, it is easily verified that

$$w_1 = \cos 2x - i \sin 2x,$$

$$w_2 = \cos 2x + i \sin 2x$$

are solutions of R_B^V.

The integrals of R_F^V. We have (§4) $v = \int w dx$. Therefore, if $2n$ is an odd integer (i.e. if g exists), resolve $\frac{x^{-1 \pm 2n}}{g^2}$ into partial fractions. Because $\int \frac{\cos 2x\, dx}{x - a}$, $\int \frac{\cos 2x\, dx}{x - a}$ cannot be expressed rationally in terms of x and $\cos x$, the resolvent R_F^V does not have an integral in the domains discussed in this section, $R[x]$, $C[x]$, $R[x, \sin x]$, $C[x, \sin x]$.

However, if $2n = \pm 1$,

$$v_1 = \cos 2x - i \sin 2x,$$

$$v_2 = \cos 2x + i \sin 2x,$$

are solutions of R_F^V.

a) *Let the domain of rationality be $R[x]$.* The resolvent R_A^V is the only one which has a rational integral and therefore the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y = 0, \qquad n = \text{any real number.}$$

b) *Let the domain of rationality be* $C[x]$.

Case 1. $2n \neq$ odd integer. The only one of the resolvents which has a rational integral is R_A^v. Therefore the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{4}{x^2}\right)y = 0.$$

Case 2. $2n =$ odd integer. The resolvent R_A^v has a rational integral and the required group is a sub-group of A.

The resolvent R_B^v has two rational integrals and (by §§3 and 4) the required group is a sub-group of D. Therefore the required group is the greatest sub-group common to A and D, i. e., the group of the equation is

$$y_1 p_1 - y_2 p_2.$$

This is the group H with $a = -b = 1$.

c) *Let the domain of rationality be* $R[x, \sin x]$. The same discussion and result as in *a*) applies.

d) *Let the domain of rationality be* $C[x, \sin x]$.

Case 1. $2n \neq$ odd integer. In this case R_A^v is the only resolvent having a rational integral, and the group of the equation is A.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{4}{x^2}\right)y = 0.$$

Case 2. $2n =$ odd integer.

R_A^v has one rational integral, giving the group A;

R_B^v has two rational integrals, giving the group D;

R_C^v has two rational integrals, giving the group C.

The greatest sub-group common to A, C, D is the group of the equation. This group is the *Identity*.

Illustrative equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{9}{x^2}\right)y = 0.$$

10. Cauchy's Equation. For the equation VI we have

$$I=\frac{4a_2-a_1^2+2a_1}{4x^2},$$

and the corresponding resolvents are

$$R_A^{VI}:\quad \Delta'+\frac{a_1}{x}\Delta=0,$$

$$R_B^{VI}:\quad u'+u^2+\frac{4a_2-a_1^2+2a_1}{4x^2}=0,$$

$$R_C^{VI}:\quad \left(\frac{-w'}{2w}\right)'+\left(\frac{-w'}{2w}\right)^2+\frac{4a_2-a_1^2+2a_1}{4x^2}=0,$$

$$R_F^{VI}:\quad \left(\frac{-v''}{2v'}\right)'+\left(\frac{-v''}{2v'}\right)^2+\frac{4a_2-a_1^2+2a_1}{4x^2}=0.$$

An integral of R_A^{VI} is

$$\Delta=\frac{1}{x^{a_1}}.$$

Two integrals of R_B^{VI} are

$$u_1=\frac{P_1}{x},\quad u_2=\frac{P_2}{x}$$

where

$$P_1=\frac{1+\sqrt{(a_1-1)^2-4a_2}}{2},\quad P_2=\frac{1-\sqrt{(a_1-1)^2-4a_2}}{2}.$$

Two integrals of R_C^{VI} are

$$w_1=\frac{1}{x^{2P_1}},\quad w_2=\frac{1}{x^{2P_2}}.$$

Two integrals of $R_F^{VI}$are

$$v_1=\frac{x^{1-2P_1}}{1-2P_1},\quad v_2=\frac{x^{1-2P_2}}{1-2P_2}.^*$$

From these integrals we determine the group of Cauchy's equation for each case that arises.

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[*] When $P_1=P_2=\frac{1}{2}$, then $v_1=v_2=\log x$.

a–b) *Let the domain of rationality be R or C.*

Case. 1 $a_1 \neq 0$, $P_1 \neq 0$, $P_2 \neq 0$. None of the resolvents have a rational integral and the group of the equation is L_{2_2}.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}\, y = 0.$$

Case 2. $a_1 = 0$, $P_1 \neq 0$, $P_2 \neq 0$. R_A^{VI} has a rational integral $\Delta = 1$ and the group of the equation is A.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}\, y = 0.$$

Case 3. $a_1 \neq 0$, $P_1 \neq 0$, $P_2 = 0$. R_B^{VI} has a rational integral $u_2 = 0$ and R_C^{VI} has a rational integral $u_2 = 1$. Therefore the group of the equation is C.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{4x^2}\, y = 0.$$

It should be noted that if $P_2 = 0$, then $P_1 = 1$. Also, case 3 is equivalent to the conditions

$$a_1 \neq 0, \quad P_1 = 0, \quad P_2 \neq 0.$$

Case 4. $a_1 = 0$, $P_1 \neq 0$, $P_2 = 0$. R_A^{VI}, R_B^{VI} and R_C^{VI} have rational integrals. The sub-group common to A and C, viz: G, is the group of the equation.

Illustrative equation: $\dfrac{d^2y}{dx^2} = 0.$

Case 4 is equivalent to the conditions $a_1 = 0$, $P_1 = 0$, $P_2 \neq 0$.

It is impossible for P_1 and P_2 to vanish simultaneously so that no further cases can arise.

c) *Let the domain of rationality be $R[x]$.*

Case 1. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 < 0$. None of the resolvents have rational integrals and the group of the equation is L_{2_2}.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{x^2}\, y = 0.$$

Case 2. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 < 0$. The group of the equation is A.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0.$$

Case 3. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. Here R_B^{VI} has a rational integral $u = \frac{1}{2x}$; R_C^{VI} has a rational integral $w = \frac{1}{x}$. The group of the equation is C.

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} + \frac{9}{16x^2}y = 0.$$

Case 4. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 = 0$. The subgroup common to A and C, viz: G, is the group of the equation. We have to consider two subcases :

4_1) $a_1 =$ odd integer. $u = \frac{y_1'}{y_1} + \frac{a_1}{2x} = \frac{1}{2x}$, whence $y_1 = x^{\frac{1-a_1}{2}}$, and since $\frac{1 - a_1}{2} =$ integer, the finite equations of G,

$$G_1: \quad \eta_1 = y_1, \quad \eta_2 = ty_1 + y_2$$

give the rationality group of the equation.

Illustrative equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0.$$

4_2) $a_1 =$ even integer. Since $\frac{1 - a_1}{2} = \frac{\text{odd integer}}{2}$, we have $y_1 = \sqrt{x^{\text{odd power}}}$.

In this case the rationality group is *complex*,* and its finite substitutions are

$$G_2: \quad \begin{array}{ll} \eta_1 = y_1, & \eta_1 = y_1 \\ \eta_2 = ty_1 + y_2; & \eta_2 = -ty_2 + y_2. \end{array}$$

Illustrative equation :

$$\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} + \frac{9}{4x^2}y = 0.$$

Case 5. $u_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 > 0$. Here u_1 and u_2 are both rational and the group of the equation is D or one of its subgroups. We have to consider two subcases.

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* Vessiot, loc. cit., p. 261, Remark I; also p. 271.

5_1). $2P_1$ and $2P_2 \neq$ integers. Since the integrals of R_C^{VI} and R_F^{VI} are both irrational, the group of the equation is D.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{4x}\frac{dy}{dx} - \frac{1}{8x^2}y = 0.$$

5_2). $2P_1$ and $2P_2 =$ integers $\neq 1$*. Since R_C^{VI} and R_F^{VI} have rational integrals, the group of the equation is F.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{2x}\frac{dy}{dx} - \frac{91}{16x^2}y = 0.$$

It cannot happen that $2P_1 =$ integer, $2P_2 \neq$ integer; so this case need not be discussed.

Case 6. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 > 0$.

6_1) $2P_1$ and $2P_2 \neq$ integers. This is 5_1) with R_A^{VI} having a rational integral. The subgroup common to D and A, viz:

H') $\qquad y_1p_1 - y_2p_2$

is the group of the equation.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0.$$

6_2). $2P_1$ and $2P_2 =$ integers. The group of the equation is the *Identity*.

Illustrative equation:

$$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0.$$

d) *Let the domain of rationality be* $C[x]$. In this domain $\sqrt{(a_1 - 1)^2 - 4a_2}$ is rational whether the quantity under the radical is positive or negative. Therefore c), case 1, becomes identical with c), case 5; and c), case 2, becomes identical with c), case 6. The discussion for the domain $C[x]$ is thus the same as the discussion for the domain $R[x]$, cases 3, 4, 5, 6.

e) *Let the domain of rationality be* $C[x, \log x]$.

Case 1. $a_1 \neq$ integer, $(a_1 - 1)^2 - 4a_2 \neq 0$. Same discussion as c), case 5.

Case 2. $a_1 =$ integer, $(a_1 - 1)^2 - 4a_2 \neq 0$. Same discussion as c), case 6.

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* $2P_1 = 1$ means $(a_1 - 1)^2 - 4a_2 = 0$; a case already discussed.

Case 3. $a_1 \neq$ integer, $(a_1-1)^2-4a_2=0$. The resolvent R_B^{VI} has one rational integral $u=1/(2x)$; the resolvent R_C^{VI} has one rational integral $w_1=1/x$, and the resolvent R_F^{VI} has one rational integral $v_1=\log x$. The sub-group common to B, C, F, that is, F, is the group of the equation.

Illustrative equation:

$$\frac{d^2y}{dx^2}+\frac{5}{2x}\frac{dy}{dx}+\frac{9}{16x^2}y=0.$$

Case 4. $a_1=$ integer, $(a_1-1)^2-4a_2=0$. The group of the equation being the sub-group common to A and F has no infinitesimal transformation.

4_1. If $a_1=$ odd integer, the group of the equation is the *Identity*.

Same illustrative equation as c) case 4_1.

4_2). If $a_1=$ even integer, the group of the equation is a finite one, composed of the two substitutions: Identity and $\eta_1=-y_1$, $\eta_2=y_2$.

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